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On the Conjecture of Meinardus on Rational Approximaton of e^x

DIETRICH BRAESS

Institut für Mathematik, Ruhr-Universität Bochum, 463 Bochum, West Germany

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This paper is concerned with uniform approximation of e^x on the interval $[-1, +1]$ by (m, n)-degree rationals, i.e., by rational functions whose numerator and denominator have degree m and n, respectively. Several years ago, Meinardus [1, p. 168] conjectured that the norm of the error function for the best approximation is asymptotically

$$\frac{m!\, n!}{2^{m+n}(m+n)!\,(m+n+1)!} \qquad \text{as} \quad m+n \to \infty. \tag{1}$$

Recently, Newman [3] has proved that the degree of approximation is indeed better than 8 times the conjectured value. Here we will establish a lower bound by applying de la Vallée-Poussin's theorem to the rational function constructed in [3]. We will show that the error function oscillates $n+m+1$ times by evaluating a winding number.

Let

$$p(z) = \int_0^\infty t^n (t+z)^m\, e^{-t}\, dt, \qquad q(z) = \int_0^\infty (t-z)^n\, t^m e^{-t}\, dt.$$

Then p/q is the (m, n)-degree Padé approximant to e^z. Following the evaluation in [3, p. 234] we get

$$\begin{aligned} q(z)\, e^z - p(z) &= \int_0^\infty (t-z)^n\, t^m e^{-t+z}\, dt - \int_0^\infty t^n (t+z)^m\, e^{-t}\, dt \\ &= \int_0^z (t-z)^n\, t^m e^{z-t}\, dt \\ &= z^{m+n+1} \int_0^1 (u-1)^n\, u^m e^{(1-u)z}\, du. \end{aligned}$$

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Hence, for $|z| \leqslant \frac{1}{2}$,

$$\begin{aligned}|q(z)\, e^z - p(z)| &\geqslant |z|^{m+n+1} \operatorname{Re} \int_0^1 (1-u)^n\, u^m e^{(1-u)z}\, du \\ &= |z|^{m+n+1} \int_0^1 (1-u)^n\, u^m e^{(1-u)\operatorname{Re} z} \cos[(1-u) \operatorname{Im} z]\, du \\ &\geqslant |z|^{m+n+1} \int_0^1 (1-u)^n\, u^m\, du\, e^{-1/2} \cos \tfrac{1}{2} \\ &\geqslant \tfrac{7}{8} e^{-1/2}\, |z|^{m+n+1} \frac{m!\, n!}{(m+n+1)!}. \qquad (2)\end{aligned}$$

Observe that this is just $7/(8e)$ times the upper bound for $|qe^z - p|$ given in [3].

Next, an upper bound for $q(z)$. $|z| \leqslant \frac{1}{2}$, is derived:

$$\begin{aligned}|q(z)| &\leqslant \int_0^\infty (t + \tfrac{1}{2})^n\, t^m e^{-t}\, dt \\ &\leqslant e^{1/2} \int_{-1/2}^\infty (t + \tfrac{1}{2})^{n+m}\, e^{-t-1/2}\, dt \\ &= e^{1/2} (m+n)!. \qquad (3)\end{aligned}$$

By combining (2) and (3) we get

$$|e^z - p(z)/q(z)| \geqslant \frac{7}{16e} \frac{2^{-m-n} m!\, n!}{(m+n)!\,(m+n+1)!}, \qquad |z| = \tfrac{1}{2}. \qquad (4)$$

Given $x \in [-1, +1]$, put $z = (x+iy)/2$ with $x^2 + y^2 = 1$. Obviously, $e^x = e^{\bar{z}} e^z$. The crucial point is Newman's detection that $R(x) = p(\bar{z})\, p(z)/[q(\bar{z})\, q(z)]$ is an (m, n)-degree rational function in the variable x.

Put $a = e^z$, $b = p(z)/q(z)$. Then the error $e^x - R(x)$ is just $\bar{a}a - \bar{b}b$. It will be treated by using the formula

$$\bar{a}a - \bar{b}b = 2 \operatorname{Re} \bar{a}(a-b) - |a-b|^2, \qquad a, b \in \mathbb{C}. \qquad (5)$$

From (4) we get the estimate for the first term

$$|e^{\bar{z}}[e^z - p/q]| \geqslant \frac{7}{16e^{3/2}} \frac{2^{-m-n} m!\, n!}{(m+n)!\,(m+n+1)!}, \qquad |z| = \tfrac{1}{2}. \qquad (6)$$

Denote by arg w the argument of the complex number w. Then

$$\arg\{e^{\bar{z}}[e^z - p(z)/q(z)]\} = \arg\{e^{-z}[e^z - p(z)/q(z)]\}$$
$$= \arg\left\{\frac{e^{-z}}{q(z)}[q(z)\,e^z - p(z)]\right\}. \tag{7}$$

For short, let $h(z)$ denote the function within the braces in (7).

Since p/q is the Padé approximation, $z = 0$ is a zero of $qe^z - p$ of multiplicity $n + m + 1$. Moreover, $q(z) \neq 0$ for $|z| \leqslant \frac{1}{2}$ is easily checked with the techniques in [3, p. 235]. Consequently, h has the winding number $n + m + 1$ for the circle $|z| = \frac{1}{2}$. Hence, when an entire circuit has been completed, $\arg(h(z))$ is increased by $(n + m + 1)\,2\pi$. The argument is increased by $(n + m + 1)\pi$ as z traverses the upper half of the circle, because $h(x)$ is real for x on the real line. It follows by the same arguments as in [1, pp. 38–39] that h attains real values on $n + m + 2$ points $z_k = (x_k + iy_k)/2$ with $+1 = x_1 > x_2 > \cdots > x_{n+m+2} = -1$ and that the sign changes between any pair of consecutive x's. The same is true for $e^{\bar{z}}[e^z - p/q]$. Referring to (5) we have

$$\min_{1 \leqslant k \leqslant n+m+2} \left| e^{\bar{z}_k} e^{z_k} - \frac{p(\bar{z}_k)\,p(z_k)}{p(\bar{z}_k)\,q(z_k)} \right|$$
$$\geqslant \min_{|z| = 1/2} 2\,|e^{\bar{z}}[e^z - p(z)/q(z)]| - \max_{|z| = 1/2} |e^z - p(z)/q(z)|^2$$
$$\geqslant \frac{7}{8e^{3/2}} \frac{2^{-m-n} m!\, n!}{(m+n)!\,(m+n+1)!} \left\{1 - \frac{\text{const}}{2^{m+n}(m+n+1)!}\right\} \tag{8}$$

From the theorem of de la Vallée-Poussin [1, p. 147] it is known that the expression in (8) is a lower bound for the distance of e^x from the (m, n)-degree rational functions. The gap between the upper bound in [2] and the lower bound is roughly a factor $e^{5/2}/[(2 - e^{1/2}) \cos \frac{1}{2}] < 40$.

If $m = n$, one gets better estimates for the constants from the result in [2].

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